* Students are advised to solve the questions of exercises in the same sequence or as directed by the **Faculty Members.** *

Mathematics Chapter - 04

Conceptual Notes for IIT-JEE/PET/Boards

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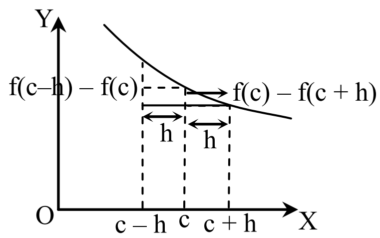
1. Definition :

1.1 Test of differentiability of y = f(x) at x = c

Let y = f(x) be defined in [a, b] and continuous in [a, b], then f(x) is said to be differentiable at x = c, a < c < b or f(x) has a derivative at x = c. If RHD = LHD

Where, RHD = $\lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$ LHD = $\lim\limits_{h\to 0}\dfrac{f(c)-f(c-h)}{h}$

RHD is called right hand derivative. LHD is called left hand derivative. Geometrically, f(x) is differentiable at a point if there exists a unique tangent at the point.



1.2 When y = f(x) is not differentiable

If LHD ≠ RHD, then f(x) is said to be not differentiable. To test the differentiability y = f(x) must be continuous. If a function is discontinuous at a point, it is not differentiable at that point.

2. Derivative of any Function :

Derivative of any function f(x) at x = a is given by : $f'(a) = \lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

(i) The right hand derivative of ' f ' at x = a

denoted by $f'_+(a)$ is defined by $f'_+(a) = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ provided the limit exists and is finite.

(ii) The left hand derivative of ' f ' at x = a

denoted by $f'_-(a)$ is defined by $f'_-(a) = \lim\limits_{h\to 0}\dfrac{f(a-h)-f(a)}{-h}$ provided the limit exists and is finite.

We also write, $f'_+(a) = f'(a^+)$ and $f'_-(a) = f'(a^-)$

(iii) Derivability and Continuity   If f ′(a) exists then f(x) is derivable at x = a ⇒ f(x) is continuous at x = a

Note : (a) Let $f'_+(a) = p$ and $f'_-(a) = q$ where, p & q are finite then

(i) p = q ⇒ f is derivable at x = a ⇒ f is continuous at x = a

(ii) p ≠ q ⇒ f is not derivable at x = a

in short, for a function f

- Differentiable ⟶ Continuous   i.e. differentiability ⇒ Continuity
- Not differentiable ⟶ May or May Not be Continuous i.e. Non differentiability ⇒ either of continuity or discontinuity
- But not continuous ⟶ Not Differentiable   i.e. Discontinuity ⇒ not differentiable

- Continuous $\longrightarrow$ may or may not differentiable i.e. continuity $\Rightarrow$ either differentiability **or** non differentiability
  (b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a

3. Differentiability over an Interval:

f(x) is said to be derivable over an interval if it is derivable at each and every point of the interval f(x) is said to be derivable over the open interval (a, b). if for any point c such that a < c < b, $f'(c^+)$ & $f'(c^-)$ exist and are equal.

Note :

i. If f(x) is differentiable at x = a & g(x) is not differentiable at x = a then the product function F(x) = f(x). g(x) can still be differentiable at x = a eg. f(x) = x and g(x) = |x|.

ii. If f(x) and g(x) both are not differentiable at x = a then the product function F(x) = f(x). g(x) can still be differentiable at x = a e.g. f(x) = |x| and g(x) = |x|.

iii. If f(x) is derivable at x = a and g(x) is not derivable at x = a, then the sum function F(x) = f(x) + g(x) must be non-derivable at x = a

iv. If f(x) and g(x) both are non-derivable at x = a then the sum function F(x) = f(x) + g(x) may be a differentiable function. e.g. f(x) = |x| and g(x) = – |x|.

4. Definition of Differential Coefficient: This is also known as differential coefficient.

If y = f(x) is a function, then derivative of y with respect to x is given by - $f'(x) = \dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

5. Different Method of Differentiation: **Differentiation by first principle** : $\dfrac{dy}{dx} = \lim\limits_{\delta x\to 0} \dfrac{\delta y}{\delta x} = \lim\limits_{\delta x\to 0} \dfrac{f(x+\delta x)-f(x)}{\delta x}$

6. Theorems of Differentiation :

(i) $\dfrac{d}{dx}$(constant) = 0

(ii) $\dfrac{d}{dx}[c\, f(x)] = c\dfrac{d}{dx}(f(x))$; where c is a constant,

(iii) $\dfrac{d}{dx}[f_1(x) \pm f_2(x)] = \dfrac{d}{dx}f_1(x) \pm \dfrac{d}{dx}f_2(x)$

(iv) $\dfrac{d}{dx}[f_1(x).f_2(x)] = f_1(x)\dfrac{d}{dx}f_2(x) + f_2(x)\dfrac{d}{dx}f_1(x)$

(v) $\dfrac{d}{dx}\left(\dfrac{f_1(x)}{f_2(x)}\right) = \left[\dfrac{f_2(x)\dfrac{d}{dx}f_1(x) - f_1(x)\dfrac{d}{dx}f_2(x)}{[f_2(x)]^2}\right]$; $f_2(x) \neq 0$

6.2 Chain rule :

If y is a function of t and t is a function of x then; $\dfrac{dy}{dx} = \dfrac{dy}{dt}.\dfrac{dt}{dx}$

In general: If y be function of z. z be function of u, u be function of v,...... Then $\dfrac{dy}{dx} = \dfrac{dy}{dz}.\dfrac{dz}{du}.\dfrac{du}{dv}$ ....

6.3 Differentiation of parametric functions:

Let x & y be the functions of parameter t,

thus x = f(t) ; y = ϕ(t) then $\dfrac{dy}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt} = \dfrac{\phi'(t)}{f'(t)}$

6.4 Differentiation by taking logarithm :

This is mostly used, for the functions where form of function is $[f(x)]^{\phi(x)}$

6.5 Differentiation of a function w.r.t to another function:

$$\frac{df(x)}{d\phi(x)} = \frac{\dfrac{d}{dx}f(x)}{\dfrac{d}{dx}\phi(x)} = \frac{f'(x)}{\phi'(x)}$$

6.6 Differentiation of implicit functions :

To find $\dfrac{dy}{dx}$ in the case of a implicit function, differentiate each term of the given equation w.r.t x, then solve the expression for the term $\dfrac{dy}{dx}$.

6.1 Some standard formulae : Some standard formulae which are used in differentiation are:

(i) $\dfrac{d}{dx}x^n = nx^{n-1}$,

(ii) $\dfrac{d}{dx}a^x = a^x \log a$

(iii) $\dfrac{d}{dx}e^x = e^x$,

(iv) $\dfrac{d}{dx}(\log x) = 1/x$ $\left(\dfrac{d}{dx}\log|x| = \dfrac{1}{x}\right)$

(v) $\dfrac{d}{dx}\cos x = -\sin x$,

(vi) $\dfrac{d}{dx}\sin x = \cos x$

(vii) $\dfrac{d}{dx}\text{cosec}\, x = -\text{cosec}\, x \cot x$

(viii) $\dfrac{d}{dx}\tan x = \sec^2 x$,

(ix) $\dfrac{d}{dx}\cot x = -\text{cosec}^2 x$

(x) $\dfrac{d}{dx}\sec x = +\sec x \tan x$

(xi) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x(\log_e a)} = \dfrac{1}{x}\log_a e$

(xii) $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$,

(xiii) $\dfrac{d}{dx}\cos^{-1}x = -\dfrac{1}{\sqrt{1-x^2}}$

(xiv) $\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{|x|\sqrt{x^2-1}}$

(xv) $\dfrac{d}{dx}\text{cosec}^{-1}x = -\dfrac{1}{|x|\sqrt{x^2-1}}$,

(xvi) $\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$

6.7 Differentiation of Determinants :

To differentiate a determinant, differentiate one row (or column) at a time, keeping others unchanged.

If $\Delta(x) = \begin{vmatrix} A(x) & B(x) \\ C(x) & D(x) \end{vmatrix}$ then $\dfrac{d}{dx}\Delta x = \begin{vmatrix} A'(x) & B'(x) \\ C(x) & D(x) \end{vmatrix} + \begin{vmatrix} A(x) & B(x) \\ C'(x) & D'(x) \end{vmatrix}$

6.8 Higher order derivatives :

If y = f(x) then the derivative of $\dfrac{dy}{dx}$ w. r. t. x is called the second derivative of y w.r.t. x and is denoted by $\dfrac{d^2y}{dx^2}$.

Similarly, $\dfrac{d^3y}{dx^3} = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$ and so on. The n^{th} **derivative** of y w.r.t x is denoted by $\dfrac{d^ny}{dx^n}$ ➢ **Note :** $\dfrac{d^2y}{dx^2} \neq \dfrac{d^2x}{dy^2}$

7. Solved Example :

Ex.1 Show that the function f(x) = |x| is continuous at x = 0. But not differentiable at x = 0.

Sol. We have $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ Since $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = 0 = f(0)$ the function is continuous at x = 0

We also have $f'(0^+) = \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x} = \lim\limits_{x\to 0} \dfrac{x-0}{x} = 1$ $f'(0^-) = \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x} = \lim\limits_{x\to 0} \dfrac{-(-x)-0}{-x} = -1$

Since, $f'(0^+) \neq f'(0^-)$, the function is not differentiable at x = 0

Ex.2 Examine differentiability of f (x) at x = 0 for $f(x) = \begin{cases} \dfrac{1-\cos x}{x\sin x}, & x \neq 0 \\ \dfrac{1}{2}, & x = 0 \end{cases}$

Sol. First we obtain $Lf'(0) = \underset{h\to 0}{Lt}\left[\dfrac{f(-h)-f(0)}{-h}\right] = \underset{h\to 0}{Lt}\left[\left(-\dfrac{1}{h}\right)\left\{\dfrac{1-\cos h}{h\sin h} - \dfrac{1}{2}\right\}\right]$

$$= \underset{h\to 0}{Lt}\left[\dfrac{h\sin h + 2(1-\cos h)}{2h^2\sin h}\right]; \left(\text{in } \dfrac{0}{0}\text{ form}\right) = \underset{h\to 0}{Lt}\left[\dfrac{h\left(h - \dfrac{h^3}{3!} + \dfrac{h^5}{5!} - \ldots\right) - 2\left(\dfrac{h^2}{2!} - \dfrac{h^4}{4!} + \dfrac{h^6}{6!} - \ldots\right)}{2h^2\left(h - \dfrac{h^3}{3!} + \ldots\right)}\right]$$

$$= \underset{h\to 0}{Lt}\left[\dfrac{h^4\left\{\left(\dfrac{1}{12} - \dfrac{1}{3!}\right) + \left(\dfrac{1}{5!} - \dfrac{2}{6!}\right)h^2\right\}}{2h^3\left(1 - \dfrac{h^2}{3!} + \ldots\right)}\right] = \underset{h\to 0}{Lt}\left[\dfrac{h\left\{\left(\dfrac{1}{12} - \dfrac{1}{3!}\right) + \left(\dfrac{1}{5!} - \dfrac{2}{6!}\right)h + \ldots\right\}}{2\left(1 - \dfrac{h^2}{3!} + \ldots\right)}\right] = 0$$

and $Rf'(0) = \underset{h\to 0}{Lt}\left(\dfrac{f(0+h)-f(0)}{h}\right) = \underset{h\to 0}{Lt}\left\{\dfrac{1}{h}\left(\dfrac{1-\cos h}{h\sin h} - \dfrac{1}{2}\right)\right\} = 0,$

Similarly, as above. i.e. Lf ' (0) = Rf ' (0) $\Rightarrow$ f (x) is differentiable at x = 0.

Ex.3 Examine differentiability of the function $f(x) = \sin^{-1}(\cos x)$ at $x = n\pi + \dfrac{\pi}{2}$ where $n \in I$

Sol. First, we obtain $Lf'\left(n\pi + \dfrac{\pi}{2}\right) = \underset{h\to 0}{Lt}\left(\dfrac{f\left(n\pi + \dfrac{\pi}{2} - h\right) - f\left(n\pi + \dfrac{\pi}{2}\right)}{-h}\right) = \underset{h\to 0}{Lt}\left[\dfrac{\sin^{-1}\left\{\cos\left(n\pi + \dfrac{\pi}{2} - h\right)\right\} - \sin^{-1}\left\{\cos\left(n\pi + \dfrac{\pi}{2}\right)\right\}}{-h}\right]$

$$= \underset{h\to 0}{Lt}\left[\dfrac{\sin^{-1}\left\{(-1)^n\cos\left(\dfrac{\pi}{2} - h\right)\right\} - \sin^{-1}\left\{(-1)^n\cos\dfrac{\pi}{2}\right\}}{-h}\right] = \underset{h\to 0}{Lt}\left[\dfrac{\sin^{-1}\{\sin(-1)^n h\} - \sin^{-1}0}{-h}\right] = \underset{h\to 0}{Lt}\dfrac{(-1)^n h}{-h} = (-1)^{n-1}$$

$$Rf'\left(n\pi + \dfrac{\pi}{2}\right) = \underset{h\to 0}{Lt}\left\{\dfrac{f\left(n\pi + \dfrac{\pi}{2} + h\right) - f\left(n\pi + \dfrac{\pi}{2}\right)}{h}\right\} = \underset{h\to 0}{Lt}\left\{\dfrac{\sin^{-1}\left[\left\{\cos\left(n\pi + \dfrac{\pi}{2} + h\right)\right\} - \sin^{-1}\left\{\cos\left(n\pi + \dfrac{\pi}{2}\right)\right\}\right]}{h}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{\sin^{-1}\left\{(-1)^n\cos\left(\frac{\pi}{2}+h\right)\right\}-\sin^{-1}\left\{(-1)^n\cos\frac{\pi}{2}\right\}}{h}\right\} = \lim_{h\to 0}\left[\frac{\sin^{-1}\{(-1)^{n+1}\sinh\}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin^{-1}\{\sin(-1)^{n+1}h\}}{h}\right] = \lim_{h\to 0}\frac{(-1)^{n+1}h}{h} = (-1)^{n+1}$$ (Which is equal to $(-1)^{n-1}$)

Thus we find $Lf'\left(n\pi+\frac{\pi}{2}\right) = Rf'\left(n\pi+\frac{\pi}{2}\right)$ $\therefore$ f (x) is differentiable at $\left(n\pi+\frac{\pi}{2}\right)$

Ex.4 Find the derivative of the function f (x), defined by f(x) = $\tan^{-1}$ x with respect to x, from the first principle.

Sol. y = $\tan^{-1}$ x or x = tan y. x + δx = tan (y + δy). Subtracting, we get δx = tan (y + δy) – tan y

$$y = \frac{\sin(y+\delta y)}{\cos(y+\delta y)}-\frac{\sin y}{\cos y} = \frac{\sin(y+\delta y)\cos y-\cos(y+\delta y)\sin y}{\cos y\cos(y+\delta y)} = \frac{\sin(\delta y)}{\cos y\cos(y+\delta y)}$$

Dividing by δy, and inverting, we have $\frac{\delta y}{\delta x} = \frac{\delta y\cos y\cos(y+\delta y)}{\sin\delta y}$

Taking limits on both sides, we obtain $\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta y\to 0}\left[\frac{\delta y}{\sin(\delta y)}\right]\lim_{\delta y\to 0}$ [cos y cos (y + δy)]

$$= 1.\cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$ Hence, we have $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

➢ **ALTER Method :** we have $\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x\to 0}\frac{\tan^{-1}(x+\delta x)-\tan^{-1}x}{\delta x}$

$$= \lim_{\delta x\to 0}\left[\frac{\tan^{-1}\left(\frac{\delta x}{1+x(\delta x+x)}\right)}{\frac{\delta x}{1+x(\delta x+x)}}\right].\frac{1}{(1+x(x+\delta x))} = \lim_{\delta x\to 0}\frac{1}{1+x(x+\delta x)} = \frac{1}{1+x^2}$$

Ex.5 Find the derivative of y with respect to x, when $y = \frac{3+u}{2+u}$, where u = $\sin^{-1}$ x,

Sol. $y = \frac{3+u}{2+u}$ where u = $\sin^{-1}$ x. We find that $\frac{dy}{du} = \left[\frac{(2+u)(1)-(3+u)(1)}{(2+u)^2}\right] = -\frac{1}{(2+u)^2}$

and $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ Hence, we obtain $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{(2+u)^2}\left[\frac{1}{\sqrt{1-x^2}}\right]$ where, u = $\sin^{-1}$ x.

Ex.6 Find $\frac{dy}{dx}$ when $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$,

Sol. (Remark: We will take help of trigonometric substitution for x so as to simplify the function). Put x = cos θ

$$\therefore y = \tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right)+\sin\left[2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right] = \tan^{-1}\left(\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}\right)+\sin\theta = \tan^{-1}\left(\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right)+\sin\theta$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\sin\theta = \frac{\pi}{4}-\frac{\theta}{2}+\sin\theta \qquad \therefore \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \left(-\frac{1}{2}+\cos\theta\right)\left(-\frac{1}{\sin\theta}\right) = \frac{(-x+1/2)}{\sqrt{1-x^2}}$$

➢**Note :** Why $\tan^{-1}\tan\frac{\theta}{2} = \frac{\theta}{2}$ & $\tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{\pi}{4}-\frac{\theta}{2}$, Our assumption x = cos θ ⇒ $0<\theta<\pi$ ⇒ $0 < \frac{\theta}{2} < \frac{\pi}{2}$

we have $\tan^{-1}\tan x = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, Also $-\frac{\pi}{4} < \frac{\pi}{4}-\frac{\theta}{2} < \frac{\pi}{4}$ Although tan $\tan^{-1}$ x = x for, ∀ x ∈ R

Ex.7 If $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+\cfrac{\sin x}{1+\cos x\ldots}}}$, prove that $\frac{dy}{dx} = \frac{(1+y)\{\cos x+\sin x\}}{1+2y+\cos x-\sin x}$

Sol. Given function is $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+y}} = \frac{(1+y)\sin x}{1+y+\cos x}$ or $y + y^2 + y\cos x = (1 + y)\sin x$

Differentiate both sides with respect to x, $\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x = (1 + y)\cos x + \left(1 + \frac{dy}{dx}\right)\sin x$

or $\frac{dy}{dx}[1 + 2y + \cos x - \sin x] = (1 + y)\cos x + y\sin x + \sin x$ or $\frac{dy}{dx} = \frac{(1+y)\{\cos x + \sin x\}}{1 + 2y + \cos x - \sin x}$

Ex.8 Find dy/dx when $x = a(\theta + \sin\theta)$ & $y = a(1 + \sin\theta)$

Sol. Differentiating with respect to θ, we obtain $\frac{dx}{d\theta} = a(1 + \cos\theta)$ and $\frac{dy}{d\theta} = a\cos\theta$.

Now, we get $\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{a\cos\theta}{a(1+\cos\theta)} = \frac{\cos\theta}{1+\cos\theta}$

Ex.9 Obtain differential coefficient of $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

Sol. Assume $u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $v = \cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

The function needs simplification before differentiation Let $x = \tan\theta$

$\therefore u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$ $\qquad v = \cos^{-1}\sqrt{\frac{1+\sec\theta}{2\sec\theta}} = \cos^{-1}\sqrt{\frac{1+\cos\theta}{2}}$

$= \cos^{-1}\left(\cos\frac{\theta}{2}\right) = \frac{\theta}{2}$ $\qquad \Rightarrow u = v \qquad \therefore \frac{du}{dv} = 1.$

Ex.10 If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + n^2y = 0$

Sol. Function is $\cos^{-1}\left(\frac{y}{b}\right) = n(\log x - \log n)$

Differentiate with respect to x, $-\frac{1}{\sqrt{1-\left(\frac{y}{b}\right)^2}}\frac{1}{b}y_1 = \frac{n}{x}$ or $\frac{-y_1}{\sqrt{b^2-y^2}} = \frac{n}{x}$ $\Rightarrow n^2(b^2 - y^2) = y_1^2x^2$

Differentiate both sides of (2) with respect to x, $-2n^2yy_1 = 2y_1y_2x^2 + 2xy_1^2$ or $x^2y_2 + xy_1 + n^2y = 0$

Ex.11 Let, $f(x + y) = f(x) + f(y) + 2xy - 1$, $\forall$ x, y $\in$ R. If f(x) is differentiable and $f'(0) = \sin\phi$, then prove that $f(x) > 0$ for $x \in R$

Sol. Put y = 0 in given relation, we get $f(x) = f(x) + f(0) - 1 \Rightarrow f(0) = 1$

We have $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $\Rightarrow f'(x) = \lim_{h\to 0}\frac{f(x)+f(h)+2hx-1-f(x)}{h}$

$f'(x) = \lim_{h\to 0}\frac{2hx}{h} + \frac{f(h)-1}{h} = 2x + \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = 2x + f'(0)$

$\Rightarrow f'(x) = 2x + \sin\phi \Rightarrow \frac{dy}{dx} = 2x + \sin\phi \Rightarrow dy = (2x + \sin\phi)\,dx \Rightarrow y = x^2 + x\sin\phi + k$

$\Rightarrow y = x^2 + x\sin\phi + 1$ $\{\because$ for x = 0, y = 1$\}$ $= \left(x + \frac{\sin\phi}{2}\right)^2 + \frac{\cos^2\phi}{4} + \frac{3}{4}$ = + ve **Hence proved.**

Ex.12 Test the differentiability of $f(x) = \tan\pi[x] + \tan x$

Sol. Clearly $\tan\pi[x] = 0\ \forall$ x $\therefore$ f(x) is actually f(x) = tan x clearly This is not differentiable at $x = \left(\frac{2n+1}{2}\right)\pi$, $n \in I$.

CBSE	Some Previous Year Question of JEE	For Class - XII

Q.1 Suppose $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$. If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$. Prove that $|a_1 + 2a_2 + ... + na_n| \le 1$.

Q.2 Let f : R → R is a function which is defined by $f(x) = \max\{x, x^3\}$ set of points on which f (x) is not differentiable is
(A) {–1, 1} (B) {–1, 0} (C) {0, 1} (D) {–1, 0, 1} **[IIT Scr. 2001]**

Q.3 Let, $\alpha \in R$, prove that a function f : R → R is differentiable at α iff there is a function g : R → R which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$. **[IIT 2001]**

Q.4 Find left hand derivative at x = k, k $\in$ I. $f(x) = [x]\sin(\pi x)$ **[IIT Scr. 2001]**
(A) $(-1)^k(k-1)\pi$ (B) $(-1)^{k-1}(k-1)\pi$ (C) $(-1)^k(k-1)k\pi$ (D) $(-1)^{k-1}(k-1)k\pi$

Q.5 Which of the following functions is differentiable at x = 0 ? **[IIT Scr. 2001]**
(A) cos (|x|) + |x| (B) cos (|x|) – |x| (C) sin (|x|) + |x| (D) sin (|x|) – |x|

Q.6 The domain of the derivative of the function f(x) = $\begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$ is- **[IIT 2002]**

(A) R – {0} (B) R – {1} (C) R – {– 1} (D) R – {– 1, 1}

Q.7 Let f(x) = $\begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$ and g(x) = $\begin{cases} x+1 & \text{if } x<0 \\ (x-1)^2+b & \text{if } x \ge 0 \end{cases}$, where a and b non-negative real numbers.

Determine the composite function gof. If (gof) (x) is continuous for all real x. Determine the values of a and b. Further, for these values of a and b, is gof differentiable at x = 0? Justify your answer. **[IIT 2002]**

Q.8 If a function f : [–2a, 2a] → R is an odd function such that f(x) = f(2a – x) for x ∈ [a, 2a] and left hand derivative at x = a is 0, then find the left hand derivative at x = – a. **[IIT 2003]**

Q.9 If f(x) is a differentiable function and f ′(2) = 6, f ′(1) = 4, f ′(c) represents the differentiation of f(x) at x = c, then

$\lim_{h\to 0} \frac{f(2+2h+h^2)-f(2)}{f(1+h^2+h)-f(1)}$ **[IIT 2003]**

(A) is equal to 3 (B) will not exist (C) may exist (D) is equal to –3

Q.10 Let y be a function of x, such that log (x + y) – 2xy = 0, then y′ (0) is- **[IIT 2004]**

(A) 0 (B) 1 (C) 1/2 (D) 3/2

Q.11 f(x) = $\begin{cases} b\sin^{-1}\left(x+\frac{c}{2}\right), & -\frac{1}{2}<x<0 \\ \frac{1}{2}, & x=0 \\ \frac{e^{\frac{ax}{2}}-1}{x}, & 0<x<\frac{1}{2} \end{cases}$ If f(x) is differentiable at x = 0 and |c| < 1/2, then find the value of a and prove that $64b^2 = \left(1-\frac{c^2}{4}\right)$ **[IIT 2004]**

Q.12 If f : [–1, 1] → R and f′(0) = $\lim_{n\to\infty} nf\left(\frac{1}{n}\right)$ and f(0) = 0 . Find the value of : $\lim_{n\to\infty} \frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right) - n$

given that $0 < \left|\lim_{n\to\infty} \cos^{-1}\left(\frac{1}{n}\right)\right| < \frac{\pi}{2}$. **[IIT 2004]**

Q.13 If two functions 'f' and 'g' satisfying given conditions for ∀ x, y ∈ R. f(x – y) = f(x) g(y) – f(y).g(x) and g(x – y) = g(x) . g(y) + f(x) f(y). If right hand derivative at x = 0 exists for f(x) then find the derivative of g(x) at x = 0.

Q.14 If x cos y + y cos x = π, then y ″ (0) = **[IIT 2005]**

(A) π (B) – π (C) 0 (D) 1

Q.15 f (x) = | |x| – 1| is not differentiable at x = **[IIT 2005]**

(A) 0, ± 1 (B) ± 1 (C) 0 (D) 1

Q.16 If f(1) = 1 ; f (2) = 4 , f (3) = 9 & f is twice differentiable then **[IIT 2005]**

(A) f ′′ (x) = 2 for atleast x ∈ [1, 3] (B) f ′′ (x) = f′ (x) = 5 ; x ∈ [1, 3] (C) f′′ (x) = 2 for only x ∈ [1 ,3] (D) f ′′ (x) = 3, for x ∈ (1, 3)

Q.17 If f is a differentiable function satisfying $f\left(\frac{1}{n}\right)$ = 0 for all n ≥ 1, n ∈ I, then- **[IIT 2005]**

(A) f(x) = 0, x ∈ (0, 1] (B) f′(0) = 0 = f(0)
(C) f(0) = 0 but f ′ (0) not necessarily zero (D) |f(x)| ≤ 1, x ∈ (0, 1]

Q.18 If f′′ (x) = –f(x) and g(x) = f ′(x) and F(x) = $\left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ & given that F(5) = 5, then F (10) is– **[IIT 2006]**

(A) 15 (B) 0 (C) 5 (D) 10

Q.19 If f(x) = min{1, x^2, x^3}, then **[IIT 2006]**

(A) f ′(x) > 0 ∀ x ∈ R (B) f(x) is continuous ∀ x ∈ R
(C) f(x) is not differentiable for two values of x (D) f(x) is not differentiable but continuous ∀ x ∈ R

Q.20 $\frac{d^2x}{dy^2}$ equal (A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ **[IIT 2007]**

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☎ आपका परिश्रम + हमारा मार्गदर्शन = निश्चित सफलता ☎

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Q.1 **(a)** If x = $\sin^{-1}$ t and y = t^3 prove that dy/dx = $3\sqrt{y(y^{1/3}-y)}$ **(b)** If y = $(1 + 1/x)^x + x^{(1 + 1/x)}$ find dy/dx.

(c) If y = $\log_u$ |cos4x| + |sin x|, where u = sec 2x find dy/dx at x = –π/6.

Q.2 **(a)** If f(x) is derivable at x = 3 and f ′(3) = 2 then, find the value of $\lim_{h\to 0} \frac{f(3+h^2)-f(3-h^2)}{2h^2}$

(b) if $f'(a) = \frac{1}{4}$ then find the value of $\lim_{h \to 0} \frac{f(a+2h^2) - f(a-2h^2)}{h^2}$

Q.3 Let R be the set of real no. & $f : R \to R$ be such that for all x & y in R, $|f(x) - f(y)| \le |x - y|^3$. Prove that f(x) is constant.

Q.4 If $f : R \to R$ is a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in R$ then show that $f(2) = f(1) - f(0)$ and find out the function.

Q.5 Find $f''(0)$ if $f(x) = 2^{\sin x} \cos(\sin x)$.

*** With Best Wishes ***